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VARIETIES OF ALGEBRAS AND ALGEBRAIC VARIETIES

BY

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To the memory of outstanding mathematician and person S. A. Amitsur

ABSTRACT

The paper contains a brief account of ideas and results, which are described in [1] and [2] with details and proofs. The subject of the paper is algebraic geometry in arbitrary algebraic structures.

1. Introduction. The main notions

1.1. CLASSICAL SITUATION. First, let us recall some well known background.

Let P be a ground field, L its extension, and $X = \{x_1, \ldots, x_n\}$ a set of variables.

Consider a ring of polynomials $P[X]$ and an affine space $L^{(n)}$. There is a Galois correspondence between subsets T in $P[X]$ and subsets A in $L^{(n)}$. If T is a subset in $P[X]$ then $T' = A$ is a set of all points $a = (a_1, \ldots, a_n)$, $a_i \in L$, which are the roots of every polynomial from T. Each A of such kind is an **algebraic variety** with the given X, P and L. If, further, A is a subset in $L^{(n)}$, then $A' = T$ is a set of all polynomials $f(x_1, \ldots, x_n)$ such that every point of A is a root of the polynomial f. The set $T = A'$ is always an ideal in $P[X]$. Let us call such an ideal a **closed** one, or, more precisely, an L-**closed ideal**. Every algebraic variety is determined by a finite set T.

There is another approach, leading to wide generalizations. Denote by Θ the variety of associative commutative algebras with unit over the field P. We call it the **classical variety**. The algebra $W = P[X]$ is the free algebra in Θ over

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the set X. The field L is also considered as an algebra in Θ. Every point $a = (a_1, \ldots, a_n)$ specifies a mapping μ: $X \to L$, $\mu(x_i) = a_i$, $i = 1, \ldots, n$. The mapping determines an algebra homomorphism μ: $W \to L$. Now we consider a point as a homomorphism, and we identify the space $L^{(n)}$ with $\mathrm{Hom}(W, L)$. A point μ is a root of a polynomial f if $f \in \mathrm{Ker}\,\mu$. The Galois correspondence can be rewritten as

$$A = T' = \{\mu |\ T \subset \mathrm{Ker}\,\mu\},$$
$$T = A' = \bigcap_{\mu \in A} \mathrm{Ker}\,\mu.$$

Now, A is a subset in the "affine space" $\mathrm{Hom}(W, L)$.

The algebraic variety generated by a set A is the set $A'' = (A')'$, and the L-closure of T is $T'' = (T')'$.

For every $L \supset P$, P being fixed, we can consider a category of algebraic varieties, denoted by $K_P(L)$. Its objects are pairs (A, X), where A is an algebraic variety with a given set of variables X. Both A and X can vary, while P and L are fixed. Morphisms will be defined in 2.1.

The principal problem arising here is to study relations between fields L_1 and L_2 which render the categories $K_P(L_1)$ and $K_P(L_2)$ isomorphic. We solve this problem in the general situation for arbitrary varieties Θ and arbitrary algebras in Θ.

Definition 1: Two extensions L_1 and L_2 of the ground field P are called **geometrically equivalent** if, for every finite X, an ideal T in $P[X]$ is L_1-closed if and only if it is L_2-closed.

It can be checked that if L_1 and L_2 are geometrically equivalent, then the corresponding categories are isomorphic. So, the problem is to consider the converse question.

If the ground field P is algebraically closed then, according to the Hilbert Nullstellensatz, every L_1 and L_2 are equivalent. In the general case, if P is not closed, the problem of equivalence of two fields is difficult. However, it can be solved in particular cases.

(1) It is easy to see that if L_1 and L_2 are P-isomorphic, then they are equivalent.

(2) If L_1 and L_2 are finite extensions of the field P, then equivalence implies isomorphism.

(3) Let L be a field, considered as an algebra over P, and $\tilde{L}$ be an ultra-power of this algebra. $\tilde{L}$ is also a field, which is an extension of the field P. Then L and $\tilde{L}$ are geometrically equivalent.

(4) A problem. When are really closed fields L_1 and L_2 geometrically equivalent?

1.2. GENERALIZATIONS. Generalizations are considered in detail in [1], but in contrast, in this paper we restrict ourselves to one-sorted algebras.

Assume that Θ is an arbitrary variety of algebras of signature Ω. It can be the classical variety, as well as a variety of groups, semigroups, etc.

Take a set of variables X, either finite or infinite, and let $W = W(X)$ be the free algebra over X in Θ.

For every algebra $G \in \Theta$ we regard a corresponding affine space as a set of points $\mathrm{Hom}(W, G)$. Equations in W have the form $w \equiv w'$, where $w, w' \in W$. A point μ satisfies this equation if the equality $w^\mu = w'^\mu$ holds in G, i.e., if the pair (w, w') lies in the kernel $\mathrm{Ker}\,\mu$. A kernel of a homomorphism is, generally speaking, a congruence in W, [5], while a set T is a binary relation in W, which consists of pairs (w, w').

We establish a Galois correspondence between relations T and subsets A in $\mathrm{Hom}(W, G)$ by the rule described above. Let us rewrite it for the new situation.

If T is a binary relation, then

$$A = T' = \{\mu | T \subset \mathrm{Ker}\,\mu\}.$$

We call A an **algebraic variety** for given Θ and G.

If A is a subset in $\mathrm{Hom}(W, G)$, then

$$T = A' = \bigcap_{\mu \in A} \mathrm{Ker}\,\mu.$$

A' is always a congruence in W. We call it G-**closed**. If A is an arbitrary set then its closure up to the variety is A'', and if T is a congruence in W then its G-closure is T''.

According to the definition, T'' is the intersection of all $\mathrm{Ker}\,\mu$ with the property $T \subset \mathrm{Ker}\,\mu$. This implies that T'' is the intersection of all congruences τ in W, containing T, such that there is an injection $W/\tau \to G$.

Let us point out one more characteristic of T''. First we introduce a special notion. Let G and H be algebras in Θ. Consider a set $\mathrm{Hom}(H, G)$. Denote by

$(G - \mathrm{Ker})(H)$ the intersection of kernels of all homomorphisms of this set. It is a congruence in H.

Let now T be a congruence in $W = W(X)$, and $\mu_0\colon W \to W/T$ be a natural homomorphism.

THEOREM 1:

$$T'' = \mu_0^{-1}(G - \mathrm{Ker})(W/T).$$

Algebras G_1 and G_2 from Θ are called X-**equivalent**, if every congruence T in $W = W(X)$ is G_1-closed if and only if it is G_2-closed.

The minimal G-closed congruence in $W(X)$ is a congruence $\mathrm{Id}_X(G)$, i.e., a congruence of identities of the algebra G in $W(X)$. If G_1 and G_2 are equivalent, then $\mathrm{Id}_X(G_1) = \mathrm{Id}_X(G_2)$. It follows from Theorem 1 that G_1 and G_2 are X-equivalent if and only if

$$(G_1 - \mathrm{Ker})(W/T) = (G_2 - \mathrm{Ker})(W/T)$$

for every congruence T in W.

Definition 2: Algebras G_1 and G_2 are called **geometrically equivalent**, if they are X-equivalent for every finite X.

It can be checked directly that an algebra G and any Cartesian power of G are geometrically equivalent.

As usual, $\mathrm{Var}\, G$ denotes the variety of algebras, generated by the algebra G.

THEOREM 2: *If G_1 and G_2 are geometrically equivalent, then* $\mathrm{Var}\, G_1 = \mathrm{Var}\, G_2$.

This means that G_1 and G_2 have the same equational theory. In particular, a commutative group cannot be equivalent to a noncommutative one.

Consider two finite sets X and Y and let

$$s\colon W(Y) \to W(X)$$

be a homomorphism of free algebras. Given G, s has a corresponding map

$$\tilde{s}\colon \mathrm{Hom}(W(X), G) \to \mathrm{Hom}(W(Y), G))$$

given by the rule: $\tilde{s}(\nu) = \nu s$ for every $\nu \in \mathrm{Hom}(W(X), G))$. Here, $\nu s(y) = \nu(s(y))$.

Define actions of a homomorphism s on the sets of points A and on the binary relations T.

Let $A \subset \mathrm{Hom}(W(X), G)$. Then $As = B = \{\nu s, \nu \in A\}$ is a subset in $\mathrm{Hom}(W(Y), G)$.

Let now $B \subset \mathrm{Hom}(W(Y), G)$. Then $A = sB$ is a subset of $\mathrm{Hom}(W(X), G)$, defined by: $\nu \in sB$ if $\nu s \in B$.

Let T be a binary relation in $W(Y)$. Then sT is a relation in $W(X)$, determined by the rule: $w(sT)w'$ if there are $w_0, w_0' \in W(Y)$ such that $s(w_0) = w$, $s(w_0') = w'$ and $w_0 T w_0'$.

If T is a relation in $W(X)$, then we define the relation Ts in $W(Y)$ as $w(Ts)w' \Leftrightarrow s(w)Ts(w')$. If T is a congruence in $W(X)$, then Ts is a congruence in $W(Y)$.

THEOREM 3:

(1) *If T is a binary relation in $W(Y)$, then*

$$(sT)' = sT.'$$

(2) *If A is a subset in* $\mathrm{Hom}(W(X), G)$, *then*

$$(As)' = A's.$$

(3) *If s is an isomorphism, $B \subset$* $\mathrm{Hom}(W(Y), G)$, *then*

$$(sB)' = sB'.$$

COROLLARY 1:

(1) *If $B = T'$ is an algebraic variety in* $\mathrm{Hom}(W(Y), G)$, *then sB is an algebraic variety in* $\mathrm{Hom}(W(X), G)$.

(2) *If $T = A'$ is a G-closed congruence in $W(X)$, then Ts is a G-closed congruence in $W(Y)$.*

(3) *If s is an isomorphism and $T = B'$ is a G-closed congruence in $W(Y)$, then sT is a G-closed congruence in $W(X)$.*

Theorem 3 is essentially used in Sections 2 and 3.

2. A category of algebraic varieties

2.1. PRELIMINARIES. For a fixed Θ and $G \in \Theta$, consider the category $K_\Theta(G)$. Its objects are algebraic varieties (A, X) where X is finite and can vary. Let us define morphisms. Let (A, X) and (B, Y) be given. Consider homomorphisms $s\colon W(Y) \to W(X)$, such that $\nu s \in B$ for any $\nu \in A$. Homomorphisms s and s' are equivalent, if $\nu s = \nu s'$ for any $\nu \in A$. Classes $\bar{s}$ of equivalent homomorphisms are morphisms $\bar{s}\colon A \to B$. Multiplication of classes is determined by multiplication of representatives. This leads to the category $K_\Theta(G)$.

Define, simultaneously, a category $\mathbb{C}_\Theta(G)$. Its objects are algebras $W(X)/T$, with G-closed congruence T in $W(X)$. Morphisms are homomorphisms of such algebras. $\mathbb{C}_\Theta(G)$ is a subcategory in Θ and, moreover, in the category $\mathrm{Var}\, G$.

THEOREM 4: *Categories $K_\Theta(G)$ and $\mathbb{C}_\Theta(G)$ are dually isomorphic.*

The passage between categories is given by

$$(A, X) \to W(X)/A'.$$

Now let Θ_0 be a subvariety in Θ and $G \in \Theta_0$. Then we can consider the category $K_{\Theta_0}(G)$.

THEOREM 5: *Categories $K_\Theta(G)$ and $K_{\Theta_0}(G)$ are canonically isomorphic.*

Thus, we can proceed from any Θ_0, and therefore the category $K_\Theta(G)$ depends up to isomorphisms of categories only on the algebra G. In particular, one can take $\Theta_0 = \mathrm{Var}\, G$. Denote $K_{\mathrm{Var}\, G}(G)$ by K_G.

2.2. SIMILARITY OF ALGEBRAS. The notion of geometrical similarity of algebras generalizes that of geometrical equivalence of algebras.

Let us start with several remarks about congruences T in free algebras $W = W(X)$, $X = \{x_1, \ldots, x_n\}$. Given T, define a relation $\rho = \rho(T)$ on the semigroup of endomorphisms $\mathrm{End}\, W$. We set $\nu \rho \nu'$, $\nu, \nu' \in \mathrm{End}\, W$, if and only if $\nu(w) T \nu'(w)$ for any $w \in W$. In fact, ρ is an equivalence on the semigroup $\mathrm{End}\, W$. Let us show that if $w_1 T w_2$, then there are w and $\nu \rho \nu'$ such that $\nu(w) = w_1$ and $\nu'(w) = w_2$. In other words, the congruence T can be restored by the relation ρ. Define ν and ν' by $\nu(x_1) = w_1$, $\nu'(x_1) = w_2$, $\nu(x_i) = \nu'(x_i) = x_i$, $i \neq 1$. Then $\nu(x) T \nu'(x)$ for every $x \in X$. This implies that $\nu(w) T \nu'(w)$ for every $w \in W$. Hence, $\nu \rho \nu'$. If we take x_1 for w, then $\nu(w) T \nu'(w)$, which represents $w_1 T w_2$.

We would like to mention that if T is a fully characteristic congruence, then $\rho = \rho(T)$ is a congruence in the semigroup $\operatorname{End} W$ and the semigroup $\operatorname{End} W/\rho$ is isomorphic to $\operatorname{End}(W/T)$.

Let us pass to the notion of similarity of two algebras G_1 and G_2 from Θ. We start with the case when $\operatorname{Var} G_1 = \operatorname{Var} G_2 = \Theta$. This case holds true if Θ is a classical variety, the field P is infinite, and L_1 and L_2 are its arbitrary extensions, i.e., $\operatorname{Var} L = \Theta$ for every L.

Consider a category $\mathbb{C}^0_\Theta$ which is a subcategory in Θ. Objects of the category $\mathbb{C}^0_\Theta$ are algebras $W(X)$ free in Θ with finite X; morphisms are homomorphisms of these algebras. Now let φ: $\mathbb{C}^0_\Theta \to \mathbb{C}^0_\Theta$ be an automorphism of this category. Assume that φ preserves dimension: if $\varphi(W(X)) = W(Y)$, then X and Y are of the same power. For the classical Θ, for a variety of groups and in some other cases this additional condition is fulfilled automatically, but sometimes it is not true, since the range of free algebras in some Θ is not unique.

Proceeding from a given φ, determine a relation between congruences in W and $\varphi(W)$ for any W.

Let ρ be an equivalence on $\operatorname{End} W$. Define $\varphi(\rho)$ on $\operatorname{End}\varphi(W)$ by:

$$\mu\varphi(\rho)\mu' \Leftrightarrow \varphi^{-1}(\mu)\rho\varphi^{-1}(\mu').$$

Now let T be a congruence in W and T^* be a congruence in $\varphi(W)$. We write $T\varphi T^*$ if $\rho(T^*) = \varphi(\rho(T))$.

Definition 3: Algebras G_1 and G_2 are similar if for an automorphism φ and every W the corresponding relation φ establishes bijection between G_1-closed congruences in W and G_2-closed congruences in $\varphi(W)$, and this bijection is well coordinated (in some natural sense, see [2]) with morphisms in $\mathbb{C}^0_\Theta$.

If G_1 and G_2 are equivalent, then they are similar. We can take a trivial automorphism for φ.

There is a situation when similarity leads to equivalence. We call an automorphism φ **inner**, if for every W there is an isomorphism s_W: $W \to \varphi(W)$ such that for every $\nu \in \operatorname{End} W$,

$$\varphi(\nu) = s_W^{-1}\nu s_W.$$

THEOREM 6: *If G_1 and G_2 are similar under inner automorphism φ, then they are equivalent.*

This theorem follows from Theorem 3.

Let us give a general definition of the notion of similarity between algebras.

Let Θ_1 and Θ_2 be two subvarieties in the variety Θ. Take categories of free algebras $\mathbb{C}^0_{\Theta_1}$ and $\mathbb{C}^0_{\Theta_2}$ for Θ_1 and Θ_2 respectively, and assume that these categories are isomorphic, with φ an isomorphism between them. Denote objects in $\mathbb{C}^0_{\Theta_1}$ by $W^1(X)$ and in $\mathbb{C}^0_{\Theta_2}$ by $W^2(Y)$. As before, we assume that if $\varphi(W^1(X)) = W^2(Y)$, then X and Y have the same power. The presence of such isomorphism φ does not mean that Θ_1 and Θ_2 coincide.

Once more we associate congruences in W^1 and $\varphi(W^1) = W^2$, and write

$$T\varphi T^* \Leftrightarrow \rho(T^*) = \varphi(\rho(T)).$$

Now let $G_1 \in \Theta_1$ and $G_2 \in \Theta_2$. These algebras are similar under the isomorphism φ if the relation φ establishes a bijection between G_1-closed congruences in W^1 and G_2-closed congruences in $W^2 = \varphi(W^1)$, for every $W^1 = W^1(X)$, and the coordination with morphisms holds.

Definition 4: Algebras G_1 and G_2 are similar if they are φ-similar under some φ between $\mathbb{C}^0_{\operatorname{Var} G_1}$ and $\mathbb{C}^0_{\operatorname{Var} G_2}$.

If algebras G_1 and G_2 are equivalent, then they are similar. As we already know, equivalence of G_1 and G_2 implies $\operatorname{Var} G_1 = \operatorname{Var} G_2$. It remains to take a trivial automorphism φ.

2.3. Isomorphism of categories. According to [1], if G_1 and G_2 are geometrically equivalent, then the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are isomorphic. Now we deal with the opposite direction. An isomorphism

$$F\colon K_\Theta(G_1) \to K_\Theta(G_2)$$

induces an isomorphism of categories

$$\mathbb{C}_{\operatorname{Var} G_1}(G_1) = \mathbb{C}_{G_1} \quad \text{and} \quad \mathbb{C}_{\operatorname{Var} G_2}(G_2) = \mathbb{C}_{G_2}.$$

Let $\Phi\colon \mathbb{C}_{G_1} \to \mathbb{C}_{G_2}$ be this induced isomorphism. Suppose that Φ satisfies the following additional conditions.

$\mathbb{C}^0_{\operatorname{Var} G_1}$ is contained in the category $\mathbb{C}_{G_1}$ as a subcategory. Similarly, $\mathbb{C}_{G_2}$ contains $\mathbb{C}^0_{\operatorname{Var} G_2}$. Recall that $\mathbb{C}^0_\Theta$ is a category of free algebras of finite range in Θ.

We assume that the functor Φ induces an isomorphism φ of categories $\mathbb{C}^0_{\mathrm{Var}\,G_1}$ and $\mathbb{C}^0_{\mathrm{Var}\,G_2}$, which preserves dimension. By the definition, $\Phi(W) = \varphi(W)$.

Assume further that $\Phi(W/T) = \varphi(W)/T^*$ always holds. Here, T is a G_1-closed congruence in W and T^* is a G_2-closed congruence in $\varphi(W)$, which depends on Φ and T. Lastly, if $\mu_T\colon W \to W/T$ is a natural homomorphism, then

$$\Phi(\mu_T)\colon \varphi(W) \to \Phi(W/T) = \varphi(W)/T^*$$

is also a natural homomorphism.

Let us motivate these conditions and then return to the definition of the category $K_\Theta(G)$. Objects of this category are the pairs (A, X); A is an algebraic variety, associated with the free algebra $W(X)$. If (B, Y) is another object, then a morphism $\alpha\colon (A, X) \to (B, Y)$ acts only on left parts and we write $\alpha\colon A \to B$. Actually, we should consider also $s\colon W(Y) \to W(X)$, inducing α, and regard a morphism as a pair (α, s). In the transition to algebras $W(Y)/B'$ and $W(X)/A'$ this means that the commutative diagram

$$\begin{array}{ccc} W(Y) & \xrightarrow{\ s\ } & W(X) \\ {\scriptstyle\mu_2}\downarrow & & \downarrow{\scriptstyle\mu_1} \\ W(Y)/B' & \xrightarrow{\ \bar{s}=\alpha\ } & W(X)/A' \end{array}$$

holds true. Here, $\bar{s}\colon W(Y)/B' \to W(X)/A'$ is a morphism in the category $\mathbb{C}_\Theta(G)$, but s, μ_1 and μ_2 are not, in general, because $W(Y)$ and $W(X)$ are not always objects of the category $\mathbb{C}_\Theta(G)$. They are objects of this category if $\mathrm{Var}\,G = \Theta$. This is the situation of classical geometry.

In order to eliminate this inconvenience in the general case, we pass from the variety Θ to the variety $\Theta_1 = \mathrm{Var}\,G$. Free algebras in Θ_1 are already objects of the category $\mathbb{C}_{\Theta_1}(G)$. Now we can consider morphisms in the category $K_{\Theta_1}(G)$ as pairs $(\alpha, s)\colon (A, X) \to (B, Y)$, for which the diagram above holds true in the category $\mathbb{C}_{\Theta_1}(G)$. Take algebras $G = G_1$, and G_2, with $\mathrm{Var}\,G_2 = \Theta_2$. Consider an isomorphism

$$F\colon K_{\Theta_1}(G_1) = K_{G_1} \to K_{\Theta_2}(G_2) = K_{G_2}.$$

Here, $F(A, X) = (B, Y)$, where X and Y have the same power. For every morphism $(\alpha, s)\colon (A_1, X_1) \to (A_2, X_2)$ in $K_{\Theta_1}(G_1)$, associate a corresponding morphism $F(\alpha, s) = (\alpha', s')\colon (B_1, Y_1) \to (B_2, Y_2)$. We have also an isomorphism

$$\Phi\colon \mathbb{C}_{\Theta_1}(G_1) \to \mathbb{C}_{\Theta_2}(G_2).$$

For the pair (α, s) we have the following diagram in $\mathbb{C}_{\Theta_1}(G_1)$:

$$\begin{array}{ccc} W^1(X_2) & \xrightarrow{s} & W^1(X_1) \\ {\scriptstyle \mu_2}\downarrow & & \downarrow{\scriptstyle \mu_1} \\ W^1(X_2)/A_2' & \xrightarrow{\alpha} & W^1(X_1)/A_1' \end{array}$$

Applying Φ, we get a new diagram:

$$\begin{array}{ccc} \Phi(W^1(X_2)) & \xrightarrow{\Phi(s)} & \Phi(W^1(X_1)) \\ {\scriptstyle \Phi(\mu_2)}\downarrow & & \downarrow{\scriptstyle \Phi(\mu_1)} \\ \Phi(W^1(X_2)) & \xrightarrow{\Phi(\alpha)} & \Phi(W^1(X_1)/A_1') \end{array}$$

It is natural that this diagram should lead to coordination for the pair (α', s'). This, in its turn, means that $\Phi(s) = s'\colon W^2(Y_2) \to W^2(Y_1)$, $\Phi(\alpha) = \alpha'$. Thus, the functor Φ transforms $\mathbb{C}^0_{\Theta_1}$ into $\mathbb{C}^0_{\Theta_2}$ and determines the isomorphism φ. Homomorphisms $\Phi(\mu_2)$ and $\Phi(\mu_1)$ must be natural, and then $\Phi(W^1/T) = \varphi(W^1)/T^*$. This explaines the conditions above.

Definition 5: An isomorphism $K_\Theta(G_1) \to K_\Theta(G_2)$ is called **correct**, if for $\Phi\colon \mathbb{C}_{G_1} \to \mathbb{C}_{G_2}$ the conditions above hold true.

3. Main results

3.1. The general cases.

THEOREM 7: *Categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are correctly isomorphic if and only if algebras G_1 and G_2 are similar.*

Using the well known association, we call a semigroup S **perfect**, if every automorphism of S is inner and is induced by an invertible element from S.

Definition 6: A variety Θ is called **perfect** if for every free algebra $W(X)$ in Θ, where X is sufficiently large, the semigroup $\operatorname{End} W(X)$ is perfect.

THEOREM 8: *Let G_1 and G_2 be algebras in Θ, $\operatorname{Var} G_1 = \operatorname{Var} G_2 = \Theta_0$, where Θ_0 is perfect. Then, if the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are correctly isomorphic, the algebras G_1 and G_2 are geometrically equivalent.*

PROBLEM 1: *For which P is the classical variety perfect? In other words the question is whether the semigroups of endomorphisms of algebras of polynomials for sufficiently large X are perfect.*

Which varieties are perfect in general? What about the variety of all groups?

Theorem 9 relates to arbitrary isomorphisms (not necessarily correct). Let a subclass Θ_0 be chosen in the variety Θ (Θ_0 may be not a variety). Consider algebras $H = W(X)/T$ in Θ; $W(X)$ is a free in Θ algebra of a finite range; T is its congruence which is G-closed under some $G \in \Theta_0$. We call such algebras Θ_0-**regular**.

Definition 7: A variety Θ is called Θ_0-**special**, if every pair of Θ_0-regular algebras H_1 and H_2 with sufficiently large sets of generators are isomorphic whenever the semigroups End H_1 and End H_2 are isomorphic.

THEOREM 9: *Let a variety Θ be special under a class Θ_0; let G_1 and G_2 be algebras in Θ_0 and suppose that the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are isomorphic. Then the algebras G_1 and G_2 are geometrically equivalent.*

The following question is of interest in the framework of this theorem.

PROBLEM 2: *Let Θ be the classical variety over a field P, and let H_1 and H_2 be two finitely generated semisimple algebras in Θ. Under which conditions does an isomorphism between the semigroups* End H_1 *and* End H_2 *imply an isomorphism between the algebras H_1 and H_2?*

In the classical case the class Θ_0 consists of all fields in Θ. Thus, all regular algebras in Θ are semisimple.

3.2. Θ-ABELIAN GROUPS. Finally, let us consider a situation when Θ is the variety of all Abelian groups. In this case isomorphisms of categories of varieties are realized by additive functors.

THEOREM 10: *If Θ is a variety of Abelian groups, and G_1 and G_2 groups in Θ, then the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are isomorphic if and only if G_1 and G_2 are geometrically equivalent.*

This theorem follows from Theorem 8 on the basis of the known result [3] and from Theorem 9 by an application of the results in [4].

The general problem is to consider for Θ a variety of modules over a commutative ring K. This problem seems to be extremely interesting. Theorem 10 treats the case when the ring K is the ring of integers.

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